An Inventory Model with Split Orders and Random Lead Times

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ABSTRACT

We consider the issue of splitting a replenishment order between two suppliers in an inventory system with uncertain demand and lead time. We present a mathematical model that fully characterizes the long-run behavior of the inventory level and serves as an enabler for performance evaluation and optimization purposes.

INTRODUCTION

The concept of order-splitting in various contexts of inventory control refers to a common scenario where a replenishment order triggered at a reorder point is split among multiple suppliers and the resulting split orders are placed simultaneously. Also known as multiple sourcing for inventory replenishment, it has received a great deal of attention in the literature due to its relevance in practical settings. (The reader is referred to Minner (2003) for an extensive review of papers in this area.)

The common theme of the existing studies is to explore circumstances under which splitting the order quantity between two or more suppliers is more advantageous than committing the entire replenishment order to one vendor only. To that end, issues such as contract prices, quantity discounts, payment conditions, quality, and lead time variability are among the most common criteria used when comparing the two alternatives. The analytical results are mixed. The order-splitting strategy is known to mitigate the adverse impact of lead-time variability on service levels, provide better quality products, and lead to lower procurement costs by taking advantage of competition among vendors. Moreover, the emerging concern over geopolitical, economical and environmental crises disrupting supply chain operations has boosted interest in this strategy as a viable means of disruptions management. The single-vendor strategy, on the other hand, is praised as the enabler for establishing long-term relationships with the sole supplier within the broader framework of supply chain management, which in turn can lead to high quality products and contract-based discounts.

This paper provides an explicit account of the mathematical formulation of an inventory model which serves as a component of on-going research on the effectiveness of order-splitting as a means of disruption management. In particular, we focus our attention on the scenario where the shortages are considered to be lost (rather than backordered which is the dominant assumption in the literature). The lost-sales case is more prevalent in competitive environments; however, analytical treatment of lost-sales inventory systems are generally proven to be more difficult than backorder systems due to a problem known as 'curse of dimensionality' (see Hadley and Whitin (1963) for an excellent discussion on this subject). Our formulation yields the exact form of the steady-state distribution of the inventory level (stock-on-hand) for a continuous-review inventory system with two identical suppliers.

INVENTORY MODEL

We consider a setting where the inventory level of a single item is continuously reviewed against a random demand which follows a compound Poisson process; i.e., demand occurs according to a homogeneous Poisson process at the rate of λ customers per unit time, and demand sizes at every occurrence epoch form a sequence of independent and identically distributed continuous random variables with cumulative distribution function $G(.)$. The stock is replenished through an (s,Q)-type control policy. That is, an order of size Q is triggered as soon as the inventory level crosses the reorder level s, provided that there is no other order outstanding. The order quantity is halved and the resulting split orders of size $Q/2$ are placed with two *identical* suppliers simultaneously. Each order is delivered after an exponentially distributed lead time with an average length of σ^{-1} has elapsed. To induce analytical tractability, it is assumed that a maximum of two orders (one with each supplier) can be outstanding at any point in time. This, coupled with the lost-sales characteristic of the model, readily implies that $0 \le s < Q/2$. (Note that the maximum number of outstanding order in such a system is $1+\lfloor s/Q \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer which is less than or equal to x.) It should also be noted that while the form of the optimal control policy in general for this class of inventory systems is unknown, (s,Q)-type policies remain very popular due to their simplicity and resemblance to commonly adopted quantized ordering policies in practice.

Let $W(t)$, $t \ge 0$ denote the stochastic process that reflects the inventory level at time t. Note that $W(t) \in [0,s+Q)$ is a non-Markovian process. Recall that an any point in time, there can be 0, 1 or 2 orders outstanding. As such, let $N(t) \in \{0,1,2\}$ represents the number of outstanding orders at time t. It can be readily verified that $W_i(t) \equiv \{W(t), N(t) = i\}$, i=0,1,2, is a Markovian process under our set of assumptions. We designate $W_i(t)$, i = 0,1,2, as the system point process for the application of level-crossing methodology (Brill (1996)). The theory of level crossings asserts that the rates at which the sample-path tracing of the system point enters and exits a particular set of states are equal. Such sample path analysis results in a set of equilibrium relations which fully characterize the steady-state behavior of the system point. Let $F_i(w)$, i=0,1,2 denote the long-run probability that the inventory level does not exceed w while the number of outstanding orders is i. Also, let $f_i(w) = dF_i(w)/dw$ at points of continuity.

Following the above principle, the model equations for $W_2(t) \in [0,s]$ are:

$$\lambda \int_{\alpha=w}^{s} \bar{G}(\alpha - w) f_2(\alpha) d\alpha + \lambda \int_{\alpha=s}^{s+Q} \bar{G}(\alpha - w) dF_0(\alpha) = 2\sigma F_2(w) \qquad w \in [0,s], \tag{1}$$

where $\bar{G}(.) = 1 - G(.)$. While the left-hand side (LHS) of (1) represents the down-crossing rate of the system point into the interval $[0,w]$ due to demand occurrences, the right-hand side (RHS) term characterizes the exit rate of the system point from the same interval as the result of the delivery of an outstanding split order. Similarly, the equations for $W_1(t) \in [0, s+Q/2)$, can be structured as

$$\lambda \int_{\alpha=w}^{s+Q/2} \bar{G}(\alpha - w) dF_1(\alpha) = \sigma F_1(w) \qquad w \in [0, Q/2), \tag{2}$$

$$\lambda \int_{\alpha=w}^{s+Q/2} [\bar{G}(\alpha - w) - \bar{G}(\alpha - Q/2)] f_1(\alpha) d\alpha + 2\sigma F_2(w - Q/2) =$$
$$\lambda \int_{\alpha=Q/2}^{w} \bar{G}(\alpha - Q/2) dF_1(\alpha) + \sigma F_1(w) \qquad w \in [Q/2,\ s + Q/2). \tag{3}$$

The final set of relations corresponds to $W_0(t) \in (s, s+Q)$ and can be written as:

$$\lambda \int_{\alpha=w}^{s+Q} [\bar{G}(\alpha - w) - \bar{G}(\alpha - s)] dF_0(\alpha) = \lambda \int_{\alpha=s}^{w} \bar{G}(\alpha - s)(\alpha) f_0(\alpha) d\alpha \quad w \in (s, Q/2), \tag{4}$$

$$\lambda \int_{\alpha=w}^{s+Q} [\bar{G}(\alpha - w) - \bar{G}(\alpha - Q/2)] dF_0(\alpha) + \sigma F_1(w - Q/2) =$$
$$\lambda \int_{\alpha=Q/2}^{w} \bar{G}(\alpha - Q/2) dF_0(\alpha) \qquad w \in [Q/2,\ Q), \tag{5}$$

$$\lambda \int_{\alpha=w}^{s+Q} [\bar{G}(\alpha - w) - \bar{G}(\alpha - Q)] f_0(\alpha) d\alpha + \sigma \int_{\alpha=Q/2}^{w-Q/2} dF_1(\alpha) =$$
$$\lambda \int_{\alpha=Q}^{w} \bar{G}(\alpha - Q) dF_0(\alpha) \qquad w \in [Q,\ s + Q). \tag{6}$$

The steady-state distribution function of $W(t)$ is defined by

$$dF(w) = \begin{cases} f_2^0 & w = 0, \\ f_2(w) + f_1(w) & w \in (0, s], \\ f_1(w) + f_0(w) & w \in (s, Q/2), \\ f_1^{Q/2} + f_0^{Q/2} & w = Q/2, \\ f_1(w) + f_0(w) & w \in (Q/2, s + Q/2), \\ f_0(w) & w \in [s + Q/2, Q), \\ f_0^Q & w = Q, \\ f_0(w) & w \in (Q + s + Q), \end{cases} \tag{7}$$

where dF(0), dF(Q/2), and dF(Q) represent mass probabilities at points 0, Q/2 and Q, respectively. We next provide a detailed procedure for solving the system of equations (1)-(6) and obtaining the various components of the functional form shown in (7).

SOLUTION PROCESS

Solving the system of integral equations (1)-(6) in general calls for numerical means of solution for an arbitrary form of G(.); i.e., the distribution function of demand size at every demand occurrence epoch. In this paper we limit our attention to $G(x) = 1\text{-}e^{-\mu x}$, $x \geq 0$ (i.e., exponential distribution) to obtain a closed form solution.

Let $\langle D \rangle \equiv$ d/d w denote the differential operator. Applying $\langle D \rangle \langle D\text{-}\mu \rangle$ to (1), (2), (3), (5), and (6) leads to:

$$(\lambda + 2\sigma) Df_2(w) - 2\sigma\mu f_2(w) = 0 \qquad w \in (0, s], \tag{8}$$

$$(\lambda + \sigma) Df_1(w) - \sigma\mu f_1(w) = 0 \qquad w \in (0, Q/2), \tag{9}$$

$$(\lambda + \sigma) Df_1(w) - 2\sigma D f_2(w - Q/2) - \sigma\mu f_1(w) + 2\sigma\mu f_2(w - Q/2) = 0 \qquad w \in (Q/2, s + Q/2), \tag{10}$$

$$\lambda Df_0(w) - \sigma D f_1(w - Q/2) + \sigma\mu f_1(w - Q/2) = 0 \qquad w \in (Q/2, Q), \tag{11}$$

and

$$\lambda Df_0(w) - \sigma D f_1(w - Q/2) + \sigma\mu f_1(w - Q/2) = 0 \qquad w \in (Q, s + Q), \tag{12}$$

respectively. The general solutions to (8) and (9) can be expressed as

$$f_2(w) = a_2 e^{\eta\mu w} \qquad w \in (0, s], \tag{13}$$

and

$$f_1(w) = a_1 e^{\gamma\mu w} \qquad w \in (0, Q/2), \tag{14}$$

respectively, where $\eta = 2\sigma/(\lambda + 2\sigma)$, $\gamma = \sigma/(\lambda + \sigma)$, and a_1 and a_2 are arbitrary constants. Substituting (13) into (10) yields:

$$D f_1(w) - \gamma\mu f_1(w) + 2\gamma\bar{\eta}\, \mu a_2 e^{\eta\mu(w - Q/2)} = 0 \qquad w \epsilon (Q/2, s + Q/2), \tag{15}$$

which has a general solution of the form

$$f_1(w) = -2a_2 e^{\eta\mu(w - Q/2)} + c_1 e^{\gamma\mu w} \qquad w \epsilon (Q/2, s + Q/2), \tag{16}$$

where $\bar{\eta} = 1 - \eta$, and c_1 is an arbitrary constant. Similarly, substituting (14) and (16) in (11) and (12), respectively, provides

$$\lambda D f_0(w) + \sigma\bar{\gamma}\mu a_1 e^{\gamma\mu(w-Q/2)} = 0 \qquad w\epsilon(Q/2, Q), \tag{17}$$

$$\lambda D f_0(w) + \sigma\bar{\gamma}\mu c_1 e^{\gamma\mu(w-Q/2)} - 2\sigma\bar{\eta}\,\mu a_2 e^{\eta\mu(w-Q)} = 0 \qquad w\epsilon(Q, s+Q), \tag{18}$$

where $\bar{\gamma} = 1-\gamma$. The general solutions to (17) and (18) are

$$f_0(w) = -a_1 e^{\gamma\mu(w-Q/2)} + c_{01} \qquad w\epsilon(Q/2, Q), \tag{19}$$

$$f_0(w) = a_2 e^{\eta\mu(w-Q/2)} - c_1 e^{\gamma\mu(w-Q/2)} + c_{02} \qquad w\epsilon(Q, s+Q), \tag{20}$$

respectively, where c_{01} and c_{02} are arbitrary constants. By the same token, multiplying both sides of (4) by $e^{-\mu w}$ and applying $< D >$ results in

$$f_0(w) = a_0 \qquad w\epsilon(s, Q/2), \tag{21}$$

where a_0 is the arbitrary constant.

The next step in the solution process is to determine the values of the all the constant terms (i.e., a_0, a_1, a_2, c_1, c_{01}, c_{02}) as well as the point mass probabilities ($f_2^0, f_1^0, f_1^{Q/2}$, $f_0^{Q/2}$ and f_0^Q) at levels 0, $Q/2$, and Q. This can be accomplished by establishing eleven linearly independent relations. To that end, setting $w=Q/2$ in (3) and (5), and setting $w=Q$ in (6) provide three relations. Seven more relations can be obtained by substituting the general solutions (13), (14), (16), (19), (20), (21) into the original balance equations (1)-(6), and comparing the coefficient of common exponential terms as well as the constant terms:

$$\frac{e^{\gamma\mu Q/2}}{\bar{\gamma}}(c_1 - a_1) - \frac{2a_2}{\bar{\eta}} + \mu f_1^{Q/2} = 0, \tag{22}$$

$$\frac{2a_2}{\bar{\eta}}\, e^{\eta\mu s} - \frac{c_1}{\bar{\gamma}}\; e^{\gamma\mu(s+Q/2)} = 0, \tag{23}$$

$$c_{01} - a_0 - \frac{a_1}{\bar{\gamma}} + \mu\, f_0^{Q/2} = 0, \tag{24}$$

$$\frac{e^{\gamma\mu Q/2}}{\bar{\gamma}}\,(a_1 - c_1) + \frac{a_2}{\bar{\eta}} - c_{01} + c_{02} + \mu\, f_0^Q = 0, \tag{25}$$

$$\frac{c_1}{\bar{\gamma}} e^{\gamma\mu(s+Q/2)} - \frac{a_2}{\bar{\eta}} e^{\eta\mu s} - c_{02} = 0, \tag{26}$$

$$f_2^0 - \frac{a_2}{\eta\mu} = 0, \tag{27}$$

$$f_1^0 - \frac{a_1}{\gamma\mu} = 0, \tag{28}$$

The normalizing condition,

$$\int_0^s dF_2(w) + \int_0^{s+\frac{Q}{2}} d\,F_1(w) \; + \int_0^{s+Q} dF_0(w) = 1, \tag{29}$$

completes the set of linear relations required to fully characterize the distribution of the inventory level. Using the general functional forms (i.e., (13), (14), (16), (19), (20), and (21)), the normalizing condition can be restructured as

$$a_0\left(\frac{Q}{2} - s\right) + c_{01}\frac{Q}{2} + c_{02}s + f_2^0 + f_1^0 + f_1^{Q/2} + f_0^{Q/2} + f_0^Q = 1. \tag{30}$$

Figure 1 depicts a sample plot of the steady-state density function, $f(w)$, of the inventory level process.

CONCLUDING REMARKS

This paper presented an exact analytical treatment of the order-splitting problem in a continuous-review inventory system with random demand and replenishment lead time which deals with two identical suppliers. The proposed model resulted in the exact functional form of the stationary density function of inventory level in various state intervals, which, in turn, led to complete characterization of the probabilistic behavior of the inventory system in steady state. Such knowledge of the system behavior is crucial in establishing performance measures (e.g., average inventory, fill rate, average cost, etc.) for optimization and/or assessment purposes. The model presented here serves as a module for a broader project which is aimed at investigating the merits of multiple sourcing as a mean of alleviating the negative impacts of potential disruptions in the supply process.

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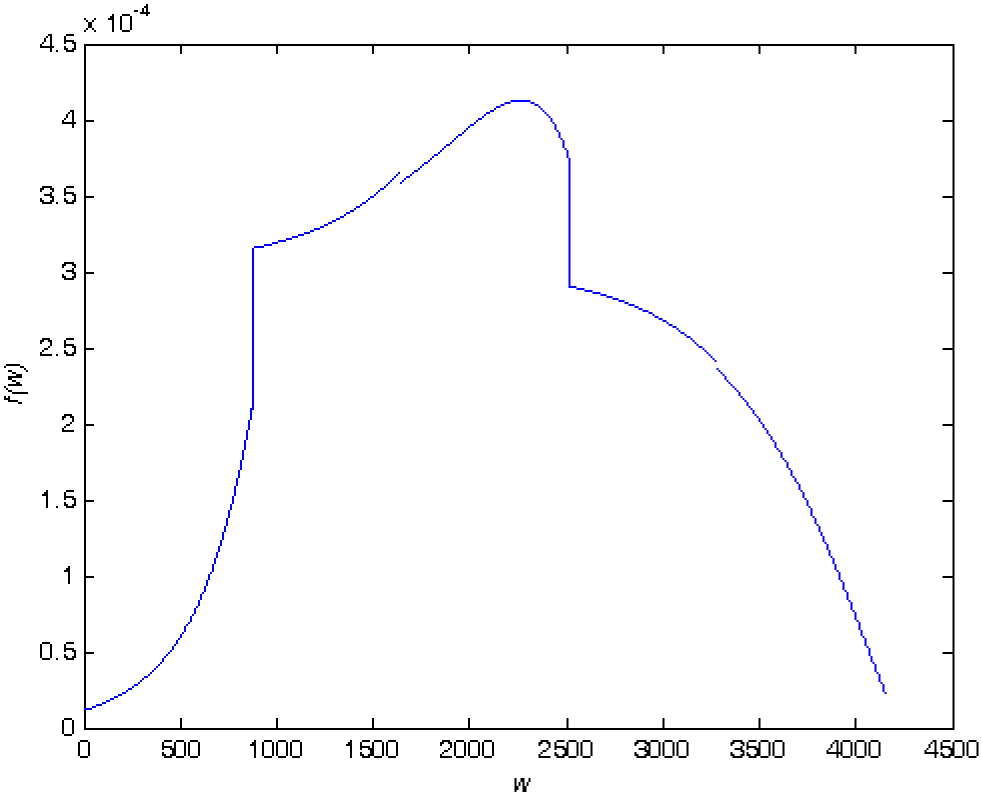

Figure 1: A typical plot of $f(w)$ for λ=300, μ=0.01, σ=80, s=872 and Q=3282.